Shadowing in CR-Dynamical Systems

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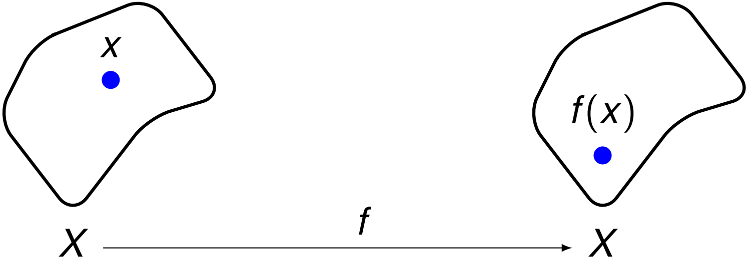
Topological Dynamical Systems

Definition

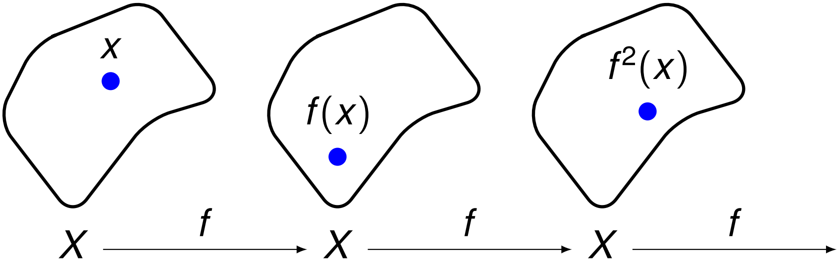
For a non-empty compact metric space X and continuous self-map $f : X \to X$, we say (X, f) is a *topological dynamical system*.



Topological Dynamical Systems

Definition

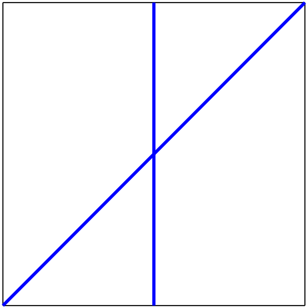
Let (X, f) be a topological dynamical system. Then the *trajectory* of a point $x \in X$ is the sequence $\langle x, f(x), f^2(x), \ldots \rangle$.



CR-Dynamical Systems

Definition

For a non-empty compact metric space X and non-empty closed $G \subseteq X \times X$, we say (X, G) is a *CR-Dynamical System*.



$X = [0, 1]$

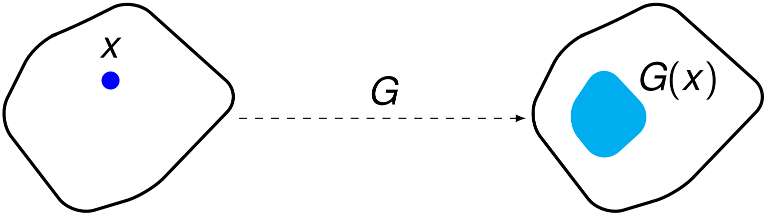
$G = \Delta_X \cup \left(\left\{ \frac{1}{2} \right\} \times X \right)$

Notation

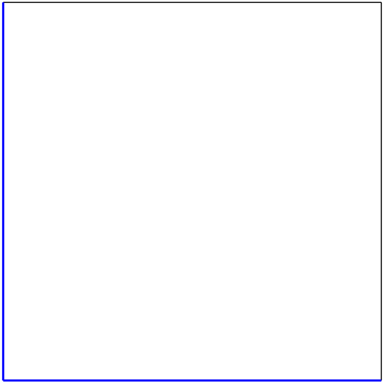
Definition

Let (X, G) be a CR-dynamical system and $x \in X$. Then we define

- $G(x) = \{y \in X \mid (x, y) \in G\}$;
- $G^n(x) = \bigcup_{y \in G^{n-1}(x)} G(y)$ for each $n > 1$; and
- $G^n = \{(x, y) \in X \times X \mid y \in G^n(x)\}$.



Example



$X = [0, 1]$

$G = X \times \{0\} \cup \{0\} \times X$

$G\left(\frac{1}{2}\right) = \{0\}$

$G^2\left(\frac{1}{2}\right) = X$

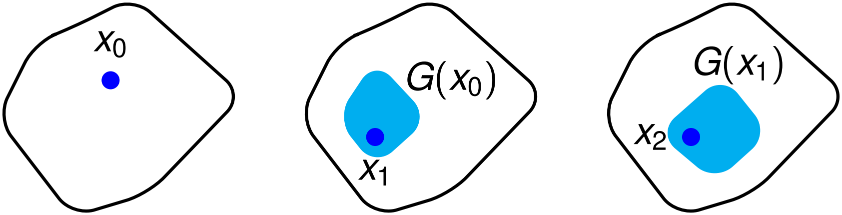
Trajectories

Definition

Let (X, G) be a CR-dynamical system. Then a *trajectory* of a point $x \in X$ is a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ such that

- $x_0 = x$; and
- $x_{n+1} \in G(x_n)$ for all $n \in \mathbb{N}$.

Denote by $T_G^+(x)$ the set of all trajectories of x.

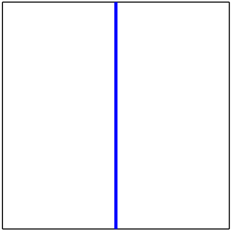


Legality

Definition

Suppose (X, G) is a CR-dynamical system.

- We say $x \in X$ is *legal* if $T_G^+(x) \neq \varnothing$
- We say $x \in X$ is *illegal* if $T_G^+(x) = \varnothing$.
- Denote by $\text{legal}(G)$ the set of legal points.



$X = [0, 1]$

$G = \{\frac{1}{2}\} \times X$

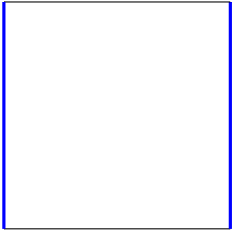
$\text{legal}(G) = \{\frac{1}{2}\}$

Domain

Definition

Suppose (X, G) is a CR-dynamical system. The *domain* of G is the set

$$\operatorname{dom}(G) = \{x \in X \mid G(x) \neq \varnothing\}.$$



$X = [0, 1]$

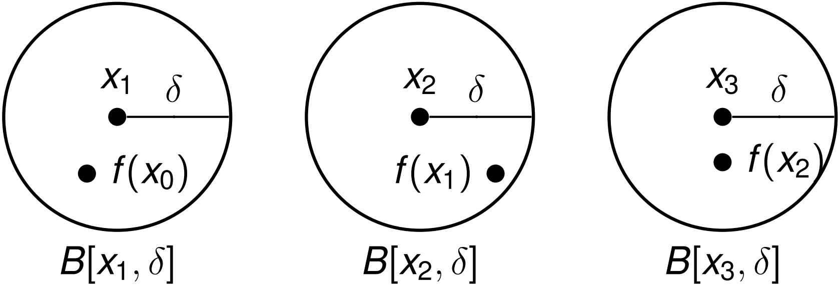
$G = \{0, 1\} \times X$

$\operatorname{dom}(G) = \{0, 1\}$

δ-Pseudo-Orbits

Definition

Let (X, f) be a topological dynamical system and $\delta > 0$. A sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ is a *δ-pseudo-orbit*, provided $d(f(x_n), x_{n+1}) \leq \delta$ for each $n \in \mathbb{N}$.



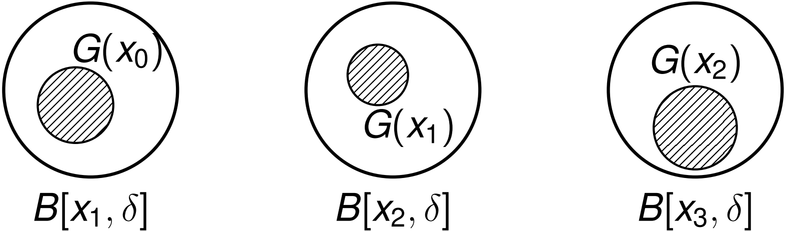
$(\delta, 1)$-Pseudo-Orbits

Definition

Let (X, G) be a CR-Dynamical System and $\delta > 0$. We say a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ in $\operatorname{dom}(G)$ is a *(δ, i)-pseudo-orbit*, provided $d(x_{n+1}, y) \le \delta$ for

- $(i = 1)$ **every** $y \in G(x_n)$,

for all $n \in \mathbb{N}$.



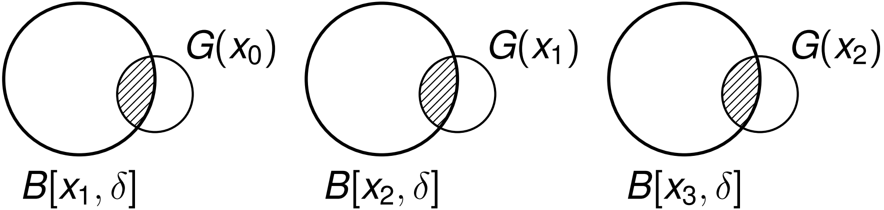
$(\delta, 2)$-Pseudo-Orbits

Definition

Let (X, G) be a CR-Dynamical System and $\delta > 0$. We say a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ in $\operatorname{dom}(G)$ is a *(δ, i)-pseudo-orbit*, provided $d(x_{n+1}, y) \leq \delta$ for

- $(i = 1)$ **every** $y \in G(x_n)$,
- $(i = 2)$ **some** $y \in G(x_n)$,

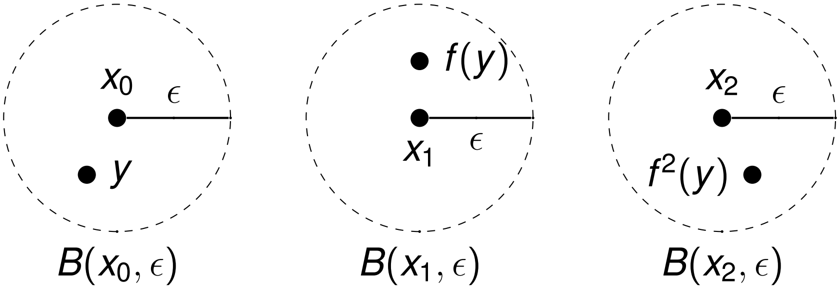
for all $n \in \mathbb{N}$.



ϵ-Shadowing-Points

Definition

Let (X, f) be a topological dynamical system and $\epsilon > 0$. A point $y \in X$ ϵ-shadows a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$, provided $d(f^n(y), x_n) < \epsilon$ for each $n \in \mathbb{N}$.



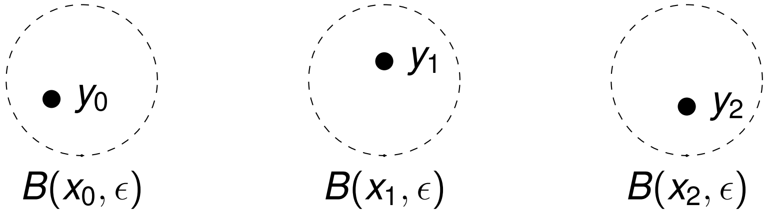
(ϵ, j)-Shadowing-Points

Definition

Let (X, G) be a CR-dynamical system and $\epsilon > 0$. We say $y \in \text{legal}(G)$ *(ϵ, j)-shadows* a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$, provided for

- $(j = 1)$ **all** $\langle y_n \mid n \in \mathbb{N} \rangle \in T_G^+(y)$,
- $(j = 2)$ **some** $\langle y_n \mid n \in \mathbb{N} \rangle \in T_G^+(y)$,

we have $d(x_n, y_n) < \epsilon$ for each $n \in \mathbb{N}$.



The Shadowing Properties

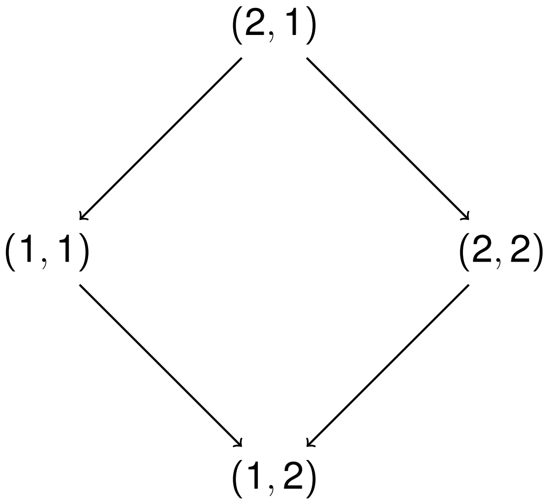
Definition

A topological dynamical system (X, f) has the *shadowing property*, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that every δ-pseudo-orbit is ϵ-shadowed by a point in X.

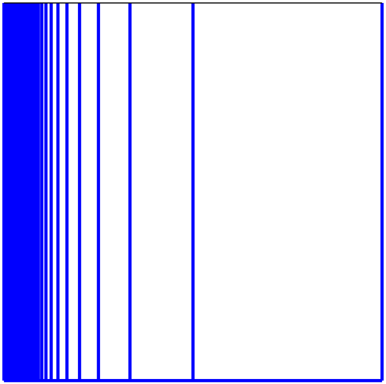
Definition

A CR-dynamical system (X, G) has the *(i, j)-shadowing property*, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that every (δ, i)-pseudo-orbit is (ϵ, j)-shadowed by a point in X.

How are they related?



The Comb Space

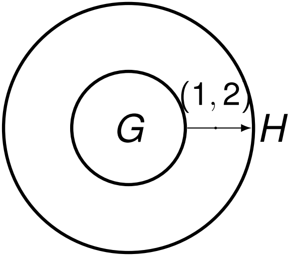


$$G = ([0,1] \times \{0\} \cup \{0\} \times [0,1]) \cup \bigcup_{n=1}^{\infty} \{\tfrac{1}{n}\} \times [0,1]$$

Extension

Theorem

Suppose (X, G) and (X, H) are CR-dynamical systems such that $G \subseteq H$ and $\text{dom}(G) = \text{dom}(H)$. If (X, G) has the $(1, 2)$-shadowing property, then (X, H) has the $(1, 2)$-shadowing property.



Generalisation

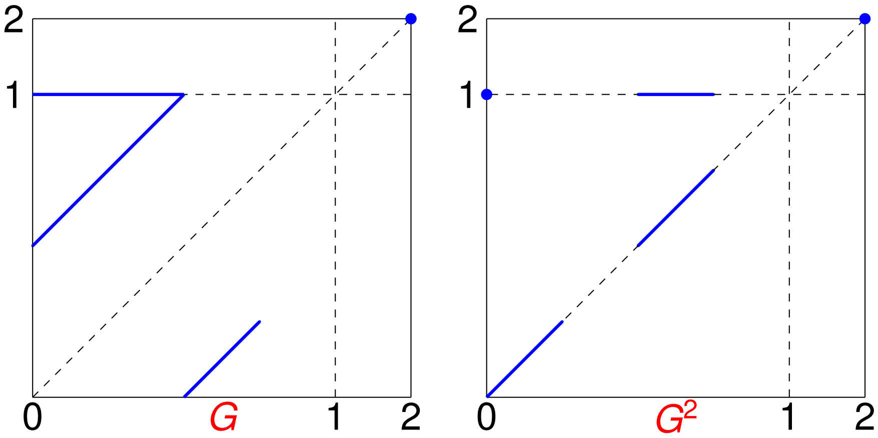
Theorem

Let (X, f) be a topological dynamical system. Then (X, f) has the shadowing property if, and only if, (X, f^n) has the shadowing property for all $n \geq 1$.

Theorem

Let (X, G) be a CR-dynamical system, and $j \in \{1, 2\}$. Then (X, G) has the $(2, j)$-shadowing property if, and only if, (X, G^n) has the $(2, j)$-shadowing property for all $n \geq 1$.

Fails for $(1, j)$-shadowing properties



Generalisation: Partial Result

Theorem

Let (X, G) be a CR-dynamical system such that $G \subseteq G^2 \subseteq G^3 \subseteq \ldots$. Then (X, G) has the $(1, 2)$-shadowing property if, and only if, (X, G^n) has the $(1, 2)$-shadowing property for all $n \geq 1$.

Proof.

$(\Longrightarrow)$ Let $k \geq 1$. By assumption $\langle G^n \mid n \geq 1 \rangle$ expands; easy to check $\mathrm{dom}\left(G^k\right) = \mathrm{legal}\left(G^k\right)$ and $\mathrm{dom}\,(G) = \mathrm{legal}\,(G)$. But also, $G \subseteq G^k$ and $\mathrm{legal}\,(G) = \mathrm{legal}\left(G^k\right)$. By Extension Theorem, $\left(X, G^k\right)$ has the $(1, 2)$-shadowing property. $\square$

Shadowing for isometries

Theorem

Let (X, f) be a topological dynamical system, such that $f : X \to X$ is an isometry. Then, (X, f) has the shadowing property if, and only if, X is totally disconnected.

Theorem

Suppose (X, G) is a CR-dynamical system, such that there exists an isometry $f : X \to X$ with $\mathrm{Graph}(f) \subseteq G$. Then,

1. *if (X, G) has the $(2, 1)$-shadowing property, then X is totally disconnected;*
2. *if X is totally disconnected, then (X, G) has the $(1, 2)$-shadowing property.*

Thank you for your attention!